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Perturbative Tamm-Dancoff Renormalization

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Abstract

A new two-step renormalization procedure is proposed. In the first step, the effects of high-energy states are considered in the conventional (Feynman) perturbation theory. In the second step, the coupling to many-body states is eliminated by a similarity transformation. The resultant effective Hamiltonian contains only interactions which do not change particle number. It is subject to numerical diagonalization. We apply the general procedure to a simple example for the purpose of illustration.

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I. INTRODUCTION

One of the most serious issues in Light-Front Field Theory (LFFT) is the complicated renormalization procedure [1]. If we wish to solve relativistic bound state problems by diagonalizing *finite* dimensional Hamiltonian matrices numerically, we are forced to make a kind of Tamm-Dancoff (TD) truncation [2], which restricts the total number of particles in any intermediate states. Although this has been very powerful in two-dimensional field theories such as the massive Schwinger model [3], it causes a serious problem in the renormalization program in higher dimensions. It is because loop diagrams contain more particles in the intermediate states and thus may not be included in the high Fock sectors in the TD approximation. On the other hand, the corresponding counterterms are not excluded by the TD truncation in a similar way. This is the source of the so-called "sector-dependent" counterterms [4].

How is it possible to solve the relativistic bound state problem with a *finite* dimensional Hamiltonian without hitting the above-mentioned problem? A field theoretical Hamiltonian is infinite dimensional in two aspects: (1) In a relativistic field theory, fluctuations of all scales couple each other. A high-energy fluctuation (or state) couples to a low-energy one in a similar way as a low-energy one does. The Hamiltonian is therefore infinite in "energy space." (2) A relativistic field theoretical Hamiltonian usually contains interactions which change the particle number of a state. If we represent the Hamiltonian in a Fock space, many-body states couple to few-body states. The Hamiltonian is infinite in "particle number space." In order to obtain an effective finite dimensional Hamiltonian matrix we have to have the control over the high-energy states and the many-body states.

Our hope is to simulate the effects of the many-body, high-energy states on the low-energy, few-body states by a set of effective interactions which do not change particle number so much and do not couple to high-energy states either. In this way, we might be able to justify the description of a bound state as a weakly bound system of the constituents. The "parton picture" of bound states is most apparent in light-front quantization. This is

precisely the reason why we are interested in LFFT.

A renormalization group approach is necessary to control the effects of high-energy and many-body states. The conventional notion of renormalization is to simulate the effects of high-energy states by a set of local operators, called counterterms. It is obvious from the discussions in the previous paragraphs that we need to generalize it so that counterterms also simulate the effects of many-body states. It is useful to distinguish two different renormalizations, though they are closely related in a complicated way. (1) "Energy renormalization" controls the effects of high-energy states. It is almost identical to the conventional one. The only difference comes from light-front quantization; in equal-time theory, a high-energy state is a high-momentum states, while on the light front, a high-energy state may have a small momentum. This implies that the counterterms in LFFT are no longer local. Nevertheless we still hope that we may use the covariant perturbation theory in this renormalization. (2) "TD renormalization" [5] deals with the effects of many-body states. Probably, the most useful scheme for the TD renormalization is the similarity transformation of the Hamiltonian [6,7]. A similarity transformation does not change the eigenvalues of the Hamiltonian. By appropriately choosing the similarity transformation, one may make the Hamiltonian more diagonal. The idea is to use similarity transformations to make the Hamiltonian more diagonal *in particle number space*.

There are two advantages in this approach: (1) We may be able to use the covariant perturbation theory in the first step. This is important because the covariant perturbation theory is very powerful in revealing the divergence structure. Furthermore, it is much easier to obtain the effects of loops than in the old-fashioned perturbation theory, or, by a perturbative similarity transformation in energy space. (2) The troubles of TD truncations are avoided at least partially. The effects of the higher Fock states are included as the effective (induced) interactions generated in the TD renormalization.

The similarity renormalization group approach is proposed by Głazek and Wilson [6] (and by Wegner [7] independently) and discussed by the Ohio-State group extensively [8]. The similarity renormalization group is designed to avoid small energy denominators in

the old-fashioned perturbation theory and to eliminate the interactions which change the *energies* of the unperturbed states drastically. There are two problems in their approach: (1) It is hard to incorporate with loop diagrams. The coupling constants do not run in the calculations so far done [8]. As far as they do not run, it is easy to obtain the so-called "coherent" Hamiltonian [9]. On the other hand, however, if one goes to higher orders and takes into account the running, one would face to the difficulty of getting it. (2) After the similarity transformation, there are still interactions which change the number of particles. In the calculations so far done, such interactions are usually ignored. In our two-step approach, we take advantage of the covariant perturbation theory to incorporate with the loop contributions in the first stage, while the interactions which change the number of particles are eliminated systematically. The TD truncation now becomes a part of the approximation, which can be improved systematically.

In this paper, we discuss a simple example to illustrate the basic idea of the two-step perturbative TD renormalization. The model is a $(1+1)$-dimensional (discretized) field theory which is a kind of hybrid of $(1+1)$-dimensional equal-time ϕ^4 theory and the one quantized on the light front. This model exhibits the mass, coupling constant, and wave function renormalization at the second-order in perturbation theory. We first obtain the counterterms in the Feynman perturbation theory. We then diagonalize the Hamiltonian in a naive Tamm-Dancoff approximation. We see that the lowest energy state is almost independent of the cutoff if the counterterms are included, while the first excited state become more cutoff dependent if the counterterms are included. This illustrates the necessity of the sector-dependent counterterms in a naive Tamm-Dancoff approximation.

Next we eliminate the interactions which change the number of particles perturbatively by a similarity transformation. The resulting effective Hamiltonian is diagonal in particle number space. We then diagonalize the effective Hamiltonian numerically. The lowest energy state is almost independent of the cutoff. It is one of our main results to show that the next-to-lowest energy state is also almost independent of the cutoff, despite that we work in a severely truncated space.

II. THE MODEL

Let us begin with the definition of the model, which we call "a^4 theory." The Hamiltonian is given by

$$H = \sum_{n=-\Lambda}^{\Lambda} \omega_n a_n^\dagger a_n + \frac{g}{6} \sum_{k,l,m,n=-\Lambda}^{\Lambda} \delta_{k,l+m+n} \left(a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k \right) + \frac{\lambda}{4} \sum_{k,l,m,n=-\Lambda}^{\Lambda} \delta_{k+l,m+n} a_k^\dagger a_l^\dagger a_m a_n \tag{2.1}$$

where "momenta" k, l, m, n are integers. The unperturbed energy ω_n is given by

$$\omega_n = |n| + \mu \tag{2.2}$$

where μ may be interpreted as mass. We assume that the cutoff Λ, a large integer, is much larger than the mass μ and the coupling constants g and λ are small enough so that the perturbation theory works well. As we will see, this model is "trivial" (see eq.(7.6)), we keep the coupling constants and the cutoff small so that we do not approach to the Landau singularity. The creation and annihilation operators are assumed to satisfy the usual commutation relations.

$$[a_n, a_m^\dagger] = \delta_{n,m}. \tag{2.3}$$

This model is similar to the usual equal-time ϕ^4 theory in $(1+1)$ dimensions but the interaction terms which contain only annihilation operators or only creation operators are absent. (This is a typical feature of LFFT.) We regard g and λ as two independent coupling constants. Unlike the light-front ϕ^4 theory, on the other hand, the "momentum" n can take negative values. The unperturbed dispersion relation is also different from the usual ones. Therefore this a^4 theory should be regarded as a hybrid model and should be studied on its own right.

The reasons why we consider this model are: (1) it is extremely simple, and (2) it exhibits mass, coupling constant, and wave function renormalization already at the second order in perturbation theory. One can study the effects of renormalization easily in this simple model.

Note that the model is invariant under the transformation $a \leftrightarrow -a$. The eigenstates are divided into the "odd" sector, which contains odd number of particles and the "even" sector, which contains even number of particles. The Fock vacuum is an eigenstate of the full Hamiltonian with the eigenvalue zero. The lowest energy state above the vacuum is the (physical) one-particle state, while the first excited state is a (physical) two-particle state.

III. FEYNMAN PERTURBATION THEORY

It is easy to develop Feynman perturbation theory for the a^4 theory, though it has no covariance. For example, the propagator is given by

$$\begin{aligned} G_{mn}(t) &\equiv \langle 0|Ta_m(t)a_n^\dagger(0)|0\rangle \\ &= \delta_{m,n}\theta(t)e^{-i\omega_n t} \\ &= \int \frac{dE}{2\pi}\frac{i\delta_{m,n}}{E-\omega_n+i\epsilon}e^{-iEt}, \end{aligned} \tag{3.1}$$

where $a_n(t)$ is the operator in the interaction picture, $a_n(t) = e^{iH_0 t}a_n e^{-iH_0 t} = a_n e^{-i\omega_n t}$, where $H_0 = \sum_{n=-\Lambda}^{\Lambda}\omega_n a_n^\dagger a_n$ is the unperturbed Hamiltonian. In the usual way, one can readily obtain the Feynman rules. (See Fig.1.)

A. self-energy

In the second-order perturbation theory, the self-energy (Fig.2-a) for the a^4 theory is given by

$$\begin{aligned} -i\Sigma_n(E) &= \frac{(-ig)^2}{3!}\sum_{p,q,r}\delta_{p+q+r,n}\int\frac{dE_p}{2\pi}\frac{dE_q}{2\pi}\frac{i}{E_p-\omega_p+i\epsilon}\frac{i}{E_q-\omega_q+i\epsilon}\frac{i}{E-E_p-E_q-\omega_r+i\epsilon} \\ &= \frac{-ig^2}{3!}\sum_{p,q,r}\delta_{p+q+r,n}\frac{1}{E-\omega_p-\omega_q-\omega_r+i\epsilon} \end{aligned} \tag{3.2}$$

where all the momenta are cutoff at $\pm\Lambda$. One can show that

$$\Sigma_n(E) = \frac{g^2}{3!}\left(-3\Lambda - \frac{3}{2}(E-3\mu)\ln\Lambda + \text{finite}\right). \tag{3.3}$$

B. coupling constant

We consider the 2-to-2 (or λ-type) interaction and the 1-to-3 (or g-type) interaction separately.

For the λ-type interaction, there are two kinds of one-loop diagrams in the second order. The first one is a *fish* diagram (Fig.2-b1),

$$\begin{aligned}\Gamma_{fish}(m,n) &= \frac{(-i\lambda)^2}{2}\sum_{p,q}\delta_{p+q,m+n}\int\frac{dE_p}{2\pi}\frac{i}{E_p-\omega_p+i\epsilon}\frac{i}{E_m+E_n-E_p-\omega_q+i\epsilon} \\ &= \frac{-i\lambda^2}{2}\sum_{p,q}\delta_{p+q,m+n}\frac{1}{E_m+E_n-\omega_p-\omega_q+i\epsilon} \\ &= \frac{-i\lambda^2}{2}(-\ln\Lambda+\text{finite}).\end{aligned} \tag{3.4}$$

The second one is an *exchange* diagram (Fig.2-b2),

$$\begin{aligned}\Gamma_{exchange}(m,k) &= \frac{(-ig)^2}{2}\sum_{p,q}\delta_{p+q,m-k}\int\frac{dE_p}{2\pi}\frac{i}{E_p-\omega_p+i\epsilon}\frac{i}{E_m-E_k-E_p-\omega_q+i\epsilon} \\ &= \frac{-ig^2}{2}\sum_{p,q}\delta_{p+q,m-k}\frac{1}{E_m-E_k-\omega_p-\omega_q+i\epsilon} \\ &= \frac{-ig^2}{2}(-\ln\Lambda+\text{finite}).\end{aligned} \tag{3.5}$$

Note that there are four similar exchange diagrams.

For the g-type interaction, there is only one kind of one-loop diagrams in the second order (Fig.2-c),

$$\begin{aligned}\Gamma_g(k,l) &= \frac{1}{2}(-ig)(-i\lambda)\sum_{p,q}\delta_{p+q,k+l}\int\frac{dE_p}{2\pi}\frac{i}{E_p-\omega_p+i\epsilon}\frac{i}{E_k+E_l-E_p-\omega_q+i\epsilon} \\ &= \frac{-ig\lambda}{2}\sum_{p,q}\delta_{p+q,k+l}\frac{1}{E_k+E_l-\omega_p-\omega_q+i\epsilon} \\ &= \frac{-ig\lambda}{2}(-\ln\Lambda+\text{finite}).\end{aligned} \tag{3.6}$$

There are three similar diagrams.

C. counterterms

In order to eliminate the divergent contributions, we add the following counterterms,

$$
\begin{aligned}
\delta H = & \sum_{n=-\Lambda}^{\Lambda} (A|n| + B) a_n^\dagger a_n \\
& + C \sum_{k,l,m,n=-\Lambda}^{\Lambda} \delta_{k,l+m+n} \left(a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k \right) \\
& + D \sum_{k,l,m,n=-\Lambda}^{\Lambda} \delta_{k+l,m+n} a_k^\dagger a_l^\dagger a_m a_n,
\end{aligned}
\tag{3.7}
$$

where

$$
A = \frac{g^2}{4} \ln \Lambda, \tag{3.8}
$$

$$
B = \frac{g^2}{2} \left(\Lambda - \mu \ln \Lambda \right), \tag{3.9}
$$

$$
C = \frac{g\lambda}{4} \ln \Lambda, \tag{3.10}
$$

$$
D = \left(\frac{\lambda^2}{8} + \frac{g^2}{2} \right) \ln \Lambda. \tag{3.11}
$$

(We are very sloppy in fixing the finite parts of the counterterms (the renormalization conditions).) Note that we do not make any rescaling (wave function renormalization) of the operators.

IV. NAIVE TAMM-DANCOFF APPROXIMATION

In this section, we discuss a naive Tamm-Dancoff approximation and the effects of the renormalization considered in the previous section. We truncate the Fock space up to including three particle states. For simplicity, we only consider the states with total momentum zero. Such a state can be expanded as

$$
\begin{aligned}
|\psi\rangle = & c a_0^\dagger |0\rangle + d \frac{(a_0^\dagger)^2}{\sqrt{2}} |0\rangle + \sum_{n=1}^{\Lambda} \psi_n a_n^\dagger a_{-n}^\dagger |0\rangle + f \frac{(a_0^\dagger)^3}{\sqrt{3!}} |0\rangle \\
& + \sum_{n=1}^{\Lambda} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min\{2n,\Lambda\}} \left(\frac{1}{\sqrt{2}} \right)^{\delta_{k,2n}} \left(\frac{1}{\sqrt{2}} \right)^{\delta_{n,2k}} \varphi_{n,k} a_n^\dagger a_{-n+k}^\dagger a_{-k}^\dagger |0\rangle,
\end{aligned}
\tag{4.1}
$$

where $\left[\frac{n+1}{2}\right]$ stands for the largest integer which is not larger than $\frac{n+1}{2}$ (Gauss' symbol). Note that the normalization condition of the state $|\psi\rangle$ is

$$\langle\psi|\psi\rangle = |c|^2 + |d|^2 + |f|^2 + \sum_{n=1}^{\Lambda} |\psi_n|^2 + \sum_{n=1}^{\Lambda} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min\{2n,\Lambda\}} |\varphi_{n,k}|^2 = 1. \tag{4.2}$$

One can diagonalize the Hamiltonian in this restricted Fock space numerically. Fig. 3-a and Fig. 3-b show the cutoff dependence of the lowest energy states with and without the counterterms. It is easy to see that the lowest state energy is almost independent of the cutoff if the counterterms are included, while the first excited state energy is almost independent if the counterterms are *not* included. This is because of the TD truncation. In this restricted Fock space, one of the particles in the two-particle sector cannot get the self-energy contribution, because the intermediate states would contain four particles and are therefore out of the restricted Fock space. If one can include, say, up to ten-particle states, the lowest energy states would not suffer from this disease because in LFFT they usually have negligibly small high Fock components. In practice, however, such a large Fock space is not feasible. It is therefore necessary to avoid this problem.

It is important to note that the Hamiltonian itself has a very simple structure. It can act on a state with any number of particles. What it does is to change the particle number by two at most. The trouble is the restriction on the *total* number of particles in a *state*. It suggests that we should restrict the *difference* of the particle numbers in a state between before and after the interaction is applied. The *difference* has no sector dependence, of course. If we can turn off the interactions which changes the particle number of a state, there is then no sector dependence. This is the basic idea of the similarity transformation in "particle space" which we discuss in the next section.

V. SIMILARITY TRANSFORMATION

Let us first consider the similarity transformation in particle space in a general setting. Consider a Hamiltonian H of the form,

$$H = H_0 + \lambda W + gV, \tag{5.1}$$

where $(H_0)_{mn} = \Omega_m \delta_{mn}$ [10]. The interaction W does not change the particle number while V does. If we includes counterterms (through the second order in the coupling constants), we will have additional terms (eq.(3.7)),

$$\delta H = g^2 H_1 + \lambda^2 W' + g^2 W'' + g\lambda V' + \cdots . \tag{5.2}$$

where W' and W'' are of λ-type and V' is of g-type and H_1 which comes from the self-energy does not change the particle number. We make a similarity transformation from H to H' by using a unitary operator $U \equiv e^{iR}$ so that $H'_{\Delta N \neq 0} = (UHU^{-1})_{\Delta N \neq 0} = 0$, where $H'_{\Delta N \neq 0}$ stands for the part of the Hamiltonian which changes the particle number. We are going to find such an operator R perturbatively. We expand R in a power series of g and λ,

$$R = gR_1 + g^2 R_2 + g\lambda T_1 + \cdots . \tag{5.3}$$

Note that R must be zero when $g = 0$. Therefore R does not contain the terms proportional only to a power of λ.

By expanding in the coupling constants, the transformed Hamiltonian H' has the form,

$$\begin{aligned} H' = {} & H_0 + g^2 H_1 + g(V + i[R_1, H_0]) + \lambda W + \lambda^2 W'' \\ & + g^2 (W'' + i[R_2, H_0] + i[R_1, V] - \frac{1}{2}[R_1, [R_1, H_0]]) \\ & + g\lambda (V' + i[T_1, H_0] + i[R_1, W]) + \cdots . \end{aligned} \tag{5.4}$$

The requirement that $H'_{\Delta N \neq 0} = 0$ determines R_1, R_2, and T_1 up to the matrix elements between the states with the same particle number. We fix this ambiguity simply by setting them zero. For example, we have

$$(R_1)_{mn} = \frac{iV_{mn}}{\Omega_n - \Omega_m}, \tag{5.5}$$

for the states m and n with different particle numbers while $(R_1)_{mn} = 0$ otherwise. In a similar way, we have

$$(R_2)_{mn} = \frac{i}{\Omega_n - \Omega_m} \left\{ W'' + i[R_1, V] - \frac{1}{2}[R_1, [R_1, H_0]] \right\}_{mn}, \tag{5.6}$$

$$(T_1)_{mn} = \frac{i}{\Omega_n - \Omega_m} \left\{ V' + i[R_1, W] \right\}_{mn}, \tag{5.7}$$

for the states m and n with different particle numbers, while $(R_2)_{mn} = (T_1)_{mn} = 0$ otherwise. The effective Hamiltonian thus takes the following form,

$$H' = H_0 + g^2 H_1 + \lambda W + \lambda^2 W' + g^2 W'' + g^2 (i[R_1, V] - \frac{1}{2}[R_1, [R_1, H_0]])_{\Delta N=0}. \tag{5.8}$$

Note that the counterterm V' does not contribute to the effective Hamiltonian through this order. This is a good thing because the corresponding diagrams are not included in the similarity transformation. See Fig.4.

VI. EFFECTIVE HAMILTONIAN

We are now ready to obtain the effective Hamiltonian for the a^4 theory. In this section, we use the notation

$$|k_1 \cdots k_n\rangle \equiv \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} a_{k_i}^\dagger |0\rangle. \tag{6.1}$$

A. one-particle sector

In the one-particle sector, the interactions W, W', and W'' in eq.(5.8) do not contribute. The Hamiltonian can be written as

$$H'_{k'k} \equiv \langle k'|H'|k\rangle = \left((1+A)|k| + \mu + B - \frac{g^2}{6} \sum_{\mathbf{l}} \frac{\delta_{k,l_1+l_2+l_3}}{\sum_{i=1}^3 \omega_{l_i} - \omega_k} \right) \delta_{k',k}, \tag{6.2}$$

where $\mathbf{l} = (l_1, l_2, l_3)$. The effective interaction (the sum) comes from the self-energy (Fig.5). For the zero momentum state, the Schrödinger equation becomes

$$E = \mu + B - \frac{g^2}{6} \sum_{\substack{p,q=-\Lambda \\ |p+q|\le\Lambda}}^{\Lambda} \frac{1}{|p| + |q| + |p+q| + 2\mu}. \tag{6.3}$$

As shown in Fig.6, the eigenvalue E is almost independent of the cutoff Λ if the counterterms are included.

B. two-particle sector

The matrix elements of the effective Hamiltonian in the two-particle sector is

$$\begin{aligned}
\langle k_1' k_2'|H'|k_1 k_2\rangle = \frac{1}{2}\Big\{&[(1+A)|k_1| + \mu + B] + [(1+A)|k_2| + \mu + B] \\
&- \frac{g^2}{6}\sum_{\mathbf{l}}\left[\frac{\delta_{k_1,l_1+l_2+l_3}}{\sum_{i=1}^3 \omega_{l_i} - \omega_{k_1}} + \frac{\delta_{k_2,l_1+l_2+l_3}}{\sum_{i=1}^3 \omega_{l_i} - \omega_{k_2}}\right]\Big\}(\delta_{k_1',k_1}\delta_{k_2',k_2} + \delta_{k_1',k_2}\delta_{k_2',k_1}) \\
&+ \frac{1}{2}\Big\{(\lambda + 4D) \\
&- \frac{g^2}{4}\sum_{p,q}[(\frac{1}{\omega_{k_1'} + \omega_p + \omega_q - \omega_{k_2}} + \frac{1}{\omega_{k_1} + \omega_p + \omega_q - \omega_{k_2'}})\delta_{k_2,k_1'+p+q} \\
&+ (\frac{1}{\omega_{k_1'} + \omega_p + \omega_q - \omega_{k_1}} + \frac{1}{\omega_{k_2} + \omega_p + \omega_q - \omega_{k_2'}})\delta_{k_1,k_1'+p+q} \\
&+ (\frac{1}{\omega_{k_2'} + \omega_p + \omega_q - \omega_{k_2}} + \frac{1}{\omega_{k_1} + \omega_p + \omega_q - \omega_{k_1'}})\delta_{k_2,k_2'+p+q} \\
&+ (\frac{1}{\omega_{k_2'} + \omega_p + \omega_q - \omega_{k_1}} + \frac{1}{\omega_{k_2} + \omega_p + \omega_q - \omega_{k_1'}})\delta_{k_1,k_2'+p+q}]\Big\}\delta_{k_1'+k_2',k_1+k_2}.
\end{aligned} \tag{6.4}$$

It is easy to see that the terms in the first braces come from the self-energy of two particles, while the ones in the second braces include the contributions from the exchange diagrams and the counterterms (Fig. 7).

A two-particle state with zero momentum can be written as

$$|\psi\rangle = \psi_0 \frac{1}{\sqrt{2}}(a_0^\dagger)^2|0\rangle + \sum_{n=1}^{\Lambda}\psi_n a_n^\dagger a_{-n}^\dagger|0\rangle. \tag{6.5}$$

By applying the effective Hamiltonian on this state, one can get the Schrödinger equation,

$$\sum_{j=0}^{\Lambda} H_{ij}\psi_j = E\psi_i \quad (i = 0, \cdots, \Lambda) \tag{6.6}$$

where

$$\begin{aligned}
H_{00} &= \frac{1}{2}\langle 0|(a_0)^2 H'(a_0^\dagger)^2|0\rangle \\
&= 2\left(\mu + B - \frac{g^2}{6}\sum_{\substack{p,q=-\Lambda \\ |p+q|\le\Lambda}}\frac{1}{|p| + |q| + |p+q| + 2\mu}\right) \\
&\quad - g^2\left(\frac{1}{2\mu} + \sum_{p=1}^{\Lambda}\frac{1}{p+\mu}\right) + \frac{1}{2}(\lambda + 4D),
\end{aligned} \tag{6.7}$$

$$
\begin{aligned}
H_{0n} = H_{n0} &= \frac{1}{\sqrt{2}}\langle 0|(a_0)^2 H' a_n^\dagger a_{-n}^\dagger|0\rangle \\
&= -\frac{g^2}{\sqrt{2}} \sum_{p=-\Lambda+n}^{\Lambda} \left[\frac{1}{n+|p|+|n-p|+2\mu} + \frac{1}{|p|+|n-p|-n+2\mu} \right] \\
&\quad + \frac{1}{\sqrt{2}}(\lambda + 4D) \qquad (6.8) \\
H_{mn} = H_{nm} &= \langle 0|a_{-m} a_m H' a_n^\dagger a_{-n}^\dagger|0\rangle \\
&= 2\left[(1+A)n + \mu + B - \frac{g^2}{6} \sum_{\substack{p,q=-\Lambda \\ |p+q-n|\le\Lambda}}^{\Lambda} \frac{1}{|p|+|q|+|n-p-q|-n+2\mu} \right] \delta_{m,n} \\
&\quad - \frac{g^2}{2} \sum_{p,q=-\Lambda}^{\Lambda} \left(\frac{1}{m+|p|+|q|-n+2\mu} + \frac{1}{n+|p|+|q|-m+2\mu} \right) \delta_{|m+p+q|,n} \\
&\quad + \lambda + 4D, \qquad (6.9)
\end{aligned}
$$

with $n, m = 1, \cdots, \Lambda$. The eigenvalues can be obtained by numerical diagonalization. The results are shown in Fig. 8. The lowest energy state in the two-particle sector corresponds to the first excited state of the whole Hamiltonian. The most important thing is that the energy of this state is almost independent of the cutoff when the counterterms are included. In other words, we have successfully renormalized the model so that the TD truncation does not cause a serious problem.

VII. DISCUSSIONS

In this paper, we have proposed a new two-step renormalization procedure and applied it to a simple model. We have obtained the effective Hamiltonian which does not contain the particle-number-changing interactions. The effective Hamiltonian is diagonalized numerically. The spectrum is almost independent of the cutoff. In this section, we discuss several aspects of our approach.

A. GWW similarity renormalization

What is the difference between the two-step approach and the Głazek-Wilson-Wegner (GWW) [6,7] similarity renormalization? Let us consider the one-particle effective interac-

tion for the a^4 theory induced by the GWW similarity renormalization,

$$\langle k'|\Delta H|k\rangle = -\frac{g^2}{6}\delta_{k',k}\sum_{\mathbf{1}}\delta_{k,l_1+l_2+l_3}\theta(\Lambda-|\Delta|)\frac{1}{\Delta}\theta(|\Delta|-\sigma), \tag{7.1}$$

where $\Delta = |l_1|+|l_2|+|l_3|-|k|+2\mu$. The sum is divergent. In the GWW similarity renormalization scheme, the counterterms are introduced to make these contributions finite. In this case, the counterterms to be added are

$$\frac{g^2}{6}\left[\frac{3}{2}\Lambda+\frac{3}{2}(|k|-2\mu)\ln\Lambda\right]\delta_{k',k}, \tag{7.2}$$

up to a finite part determined by the renormalization condition. Note that because of the different cutoff of the sum, the counterterms are similar to ours but different.

Now, in the standard OSU approach, one invokes "coupling coherence" [9]. In the present case, it is easy to obtain the so-called coherent Hamiltonian,

$$\langle k'|H_\sigma^{coh}|k\rangle = \left\{|k|+\mu+\lim_{\Lambda\to\infty}\left[A|k|+B'-\frac{g^2}{6}\sum_{\substack{p,q=-\infty\\ \sigma\le|\Delta_{p,q}|\le\Lambda}}^{\infty}\frac{1}{\Delta_{p,q}}\right]\right\}\delta_{k',k}, \tag{7.3}$$

where $\Delta_{p,q}=|p|+|q|+|k-p-q|-|k|+2\mu$, $B'=\frac{g^2}{4}(\Lambda-2\mu\ln\Lambda)$. The parameter σ should be considered as the renormalization scale. It is obvious that this is essentially the same as eq.(6.2), apart from the choice of the cutoff.

The effective two-particle interactions induced by GWW similarity renormalization are more complicated. To the second order, the induced interactions obtained by lowering the cutoff from Λ to σ are

$$\begin{aligned}
\langle k_1'k_2'|\Delta H|k_1k_2\rangle = &-\delta_{k_1'+k_2',k_1+k_2}\frac{\lambda^2}{4}\sum_{l_1,l_2}\delta_{l_1+l_2,k_1+k_2}\Big\{\theta(|\Delta^2|-\sigma)\frac{1}{\Delta^2}\theta(|\Delta^2|-|\Delta^{2'}|)\\
&+\theta(|\Delta^{2'}|-\sigma)\frac{1}{\Delta^{2'}}\theta(|\Delta^{2'}|-|\Delta^2|)\Big\}\theta(\Lambda-|\Delta^2|)\theta(\Lambda-|\Delta^{2'}|)\\
&-\Big\{\sum_{p,q,r}\delta_{k_1,p+q+r}\theta(|\Delta_1|-\sigma)\frac{1}{\Delta_1}\theta(\Lambda-|\Delta_1|)\\
&+\sum_{p,q,r}\delta_{k_2,p+q+r}\theta(|\Delta_2|-\sigma)\frac{1}{\Delta_2}\theta(\Lambda-|\Delta_2|)\Big\}(\delta_{k_1',k_1}\delta_{k_2',k_2}+\delta_{k_1',k_2}\delta_{k_2',k_1})\\
&-\delta_{k_1'+k_2',k_1+k_2}\frac{g^2}{4}\sum_{p,q}\Big\{\delta_{k_1'+p+q,k_2}\Big(\theta(|\Delta_{1'2}|-\sigma)\frac{1}{\Delta_{1'2}}\theta(|\Delta_{1'2}|-|\Delta_{12'}|)
\end{aligned}$$

$$
\begin{aligned}
&+\ \theta(|\Delta_{12'}|-\sigma)\frac{1}{\Delta_{12'}}\theta(|\Delta_{12'}|-|\Delta_{1'2}|)\Big)\,\theta(\Lambda-|\Delta_{1'2}|)\theta(\Lambda-|\Delta_{12'}|)\\
&+\delta_{k_1'+p+q,k_1}\Big(\theta(|\Delta_{1'1}|-\sigma)\frac{1}{\Delta_{1'1}}\theta(|\Delta_{1'1}|-|\Delta_{22'}|)\\
&+\ \theta(|\Delta_{22'}|-\sigma)\frac{1}{\Delta_{22'}}\theta(|\Delta_{22'}|-|\Delta_{1'1}|)\Big)\,\theta(\Lambda-|\Delta_{1'1}|)\theta(\Lambda-|\Delta_{22'}|)\\
&+\delta_{k_2'+p+q,k_2}\Big(\theta(|\Delta_{2'2}|-\sigma)\frac{1}{\Delta_{2'2}}\theta(|\Delta_{2'2}|-|\Delta_{11'}|)\\
&+\ \theta(|\Delta_{11'}|-\sigma)\frac{1}{\Delta_{11'}}\theta(|\Delta_{11'}|-|\Delta_{2'2}|)\Big)\,\theta(\Lambda-|\Delta_{11'}|)\theta(\Lambda-|\Delta_{2'2}|)\\
&+\delta_{k_2'+p+q,k_1}\Big(\theta(|\Delta_{2'1}|-\sigma)\frac{1}{\Delta_{2'1}}\theta(|\Delta_{2'1}|-|\Delta_{21'}|)\\
&+\ \theta(|\Delta_{21'}|-\sigma)\frac{1}{\Delta_{21'}}\theta(|\Delta_{21'}|-|\Delta_{2'1}|)\Big)\,\theta(\Lambda-|\Delta_{2'1}|)\theta(\Lambda-|\Delta_{21'}|)\Big\}
\end{aligned}
\tag{7.4}
$$

where $\Delta^2=|l_1|+|l_2|-|k_1|-|k_2|$, $\Delta^{2'}=|l_1|+|l_2|-|k_1'|-|k_2'|$, $\Delta_i=|p|+|q|+|r|-|k_i|+2\mu$, $\Delta_{i'j}=|p|+|q|+|k_i'|-|k_j|+2\mu$, and $\Delta_{ij'}=|p|+|q|+|k_i|-|k_j'|+2\mu$ with $i,j=1,2$.

The counterterms necessary to make the matrix elements eq.(7.4) finite are

$$
\begin{aligned}
\langle k_1'k_2'|\delta H_{c.t.}|k_1k_2\rangle=\delta_{k_1'+k_2',k_1+k_2}&\left\{\frac{\lambda^2}{4}+g^2\right\}\ln\Lambda+\frac{g^2}{12}(\delta_{k_1',k_1}\delta_{k_2',k_2}+\delta_{k_1',k_2}\delta_{k_2',k_1})\\
&\times\left\{\left[\frac{3}{2}\Lambda+\frac{3}{2}(|k_1|-2\mu)\ln\Lambda\right]+\left[\frac{3}{2}\Lambda+\frac{3}{2}(|k_2|-2\mu)\ln\Lambda\right]\right\}.
\end{aligned}
\tag{7.5}
$$

Although the structure is similar, this effective Hamiltonian is more complicated than that of eq.(6.4). This is mainly because of the cutoff employed in the GWW similarity renormalization which introduces a lot of theta functions. (If one uses a more sophisticated smeared function, the expression would contains integrations.) The point is that one has to examine the divergence structure in this complicated expression in order to obtain the counterterms.

Nevertheless, it seems obvious that the resultant "coherent" Hamiltonian is very close to that of eq.(6.4).

The real differences between the two approaches are: (1) There are still effective induced interactions, together with the canonical one, which *change* the particle number by two in the GWW approach (if $\sigma\geq 2\mu$). To restrict the Fock space to the two-particle space is to throw away the interactions. (2) The treatment of the λ-type interactions is different. In the GWW approach, the counterterms for the λ-type interactions are introduced to make the

induced effective interactions finite. On the other hand, the counterterms in the two-step approach are obtained in the first step altogether while the λ-type interactions are kept intact in the second step. There is no *fish* effective interaction. See Fig. 7. (3) In the GWW approach, only the matrix elements between the states of similar energies survive, while in the two-step approach, any matrix elements between the states of the same particle number survive. This third point may cause a non-perturbative difference, which is discussed in the next subsection.

The GWW similarity renormalization is elegant in the sense that it generates an effective Hamiltonian and the counterterms in one shot. But in practice, our two-step renormalization procedure is easier and more systematic.

It is important to see how the "vanishing energy denominators" are avoided in the two-step renormalization. The problem of "vanishing energy denominators" is fake in the sense that it happens because of our poor choice of the unperturbed states. One should instead invoke the degenerate perturbation theory. In the Feynman perturbation theory, on the other hand, this never becomes a problem because in calculating Green functions, energies (or, more appropriately, "frequencies") can be kept off mass-shell.

B. non-perturbative divergences

A difficult problem in the two-step approach is the non-perturbative divergences which arise from the λ-type interactions. In the a^4 theory, the chain diagrams (Fig. 9-a) are generated upon diagonalization of the effective Hamiltonian in the two-particle sector. They diverge as $(\frac{1}{2}\lambda \ln \Lambda)^n$, where n is the number of chains. This type of non-perturbative divergences can be canceled if

$$\lambda_\Lambda = \frac{\lambda}{1 - \frac{\lambda}{2} \ln \Lambda} \tag{7.6}$$

is used instead of $\lambda_B \equiv \lambda + 4D$. Furthermore, we have ladder diagrams for the exchange interactions (Fig. 9-b). Upon diagonalization, a complicated mixture of these diagrams occur. We need the counterterms to cancel the divergence.

Within the present approximation, the coupling constant g does not run and the one-loop renormalization group equation for λ,

$$\Lambda \frac{d\lambda_\Lambda}{d\Lambda} = \frac{\lambda_\Lambda^2}{2} + 2g^2, \tag{7.7}$$

can be solved easily. The solution is

$$\lambda_\Lambda = 2g \tan\left(\tan^{-1}\left(\frac{\lambda}{2g}\right) + g \ln \Lambda\right). \tag{7.8}$$

Although it improves the Λ-independence of the energies of two-particle states, its validity is still restricted by the perturbative nature. (The equation (7.7) is derived at the one-loop level.)

In the GWW approach, this kind of non-perturbative divergences does not seem to occur because the λ-type interaction is replaced with effective interactions which have matrix elements only between the states of similar energies. This difference is not crucial, however. In the two-step approach, one may eliminate the Hamiltonian matrix elements between the states of very different energies by utilizing the ambiguity of R_{mn} for the states m and n which have the same number of particles. Alternatively, one may make a further similarity transformation for the effective Hamiltonian in the energy space.

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[5] As we will see, this second step has little to do with divergences; they are extracted in the first step. The word "renormalization" is used in its broadest meaning that the effects of the states which are neglected are simulated by a set of effective interactions.

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[10] In this section, m and n stand for states and Ω_m for the unperturbed energy of the state m.

FIGURES

FIG. 1. Feynman rules for the a^4 theory.

FIG. 2. Diagrams in the second order in perturbation theory. There are four similar diagrams for (b2) and three for (c).

FIG. 3. The results in a naive TD approximation. The Λ-dependence of the energy eigenvalue for the lowest energy state is shown for the cases with and without counterterms in Fig. 3-a. The calculations are done for $\mu = 1.0$ and $g = \lambda = 0.01$. The same for the first excited state in Fig. 3-b.

FIG. 4. Examples of the excluded diagrams which are relevant to the coupling constant (g) renormalization.

FIG. 5. Diagrammatic representation of the effective Hamiltonian in the one-particle sector. The dot stands for counterterms.

FIG. 6. The Λ-dependence of the energy eigenvalue for the lowest energy state is shown for the cases with and without counterterms. The calculations are done for $\mu = 1.0$ and $g = \lambda = 0.01$.

FIG. 7. Diagrammatic representation of the effective Hamiltonian in the two-particle sector. The dots stand for counterterms.

FIG. 8. The Λ-dependence of the energy eigenvalue for the first excited state is shown for the cases with and without counterterms. The calculations are done for $\mu = 1.0$ and $g = \lambda = 0.01$.

FIG. 9. Examples of diagrams which lead to non-perturbative divergences on diagonalization.

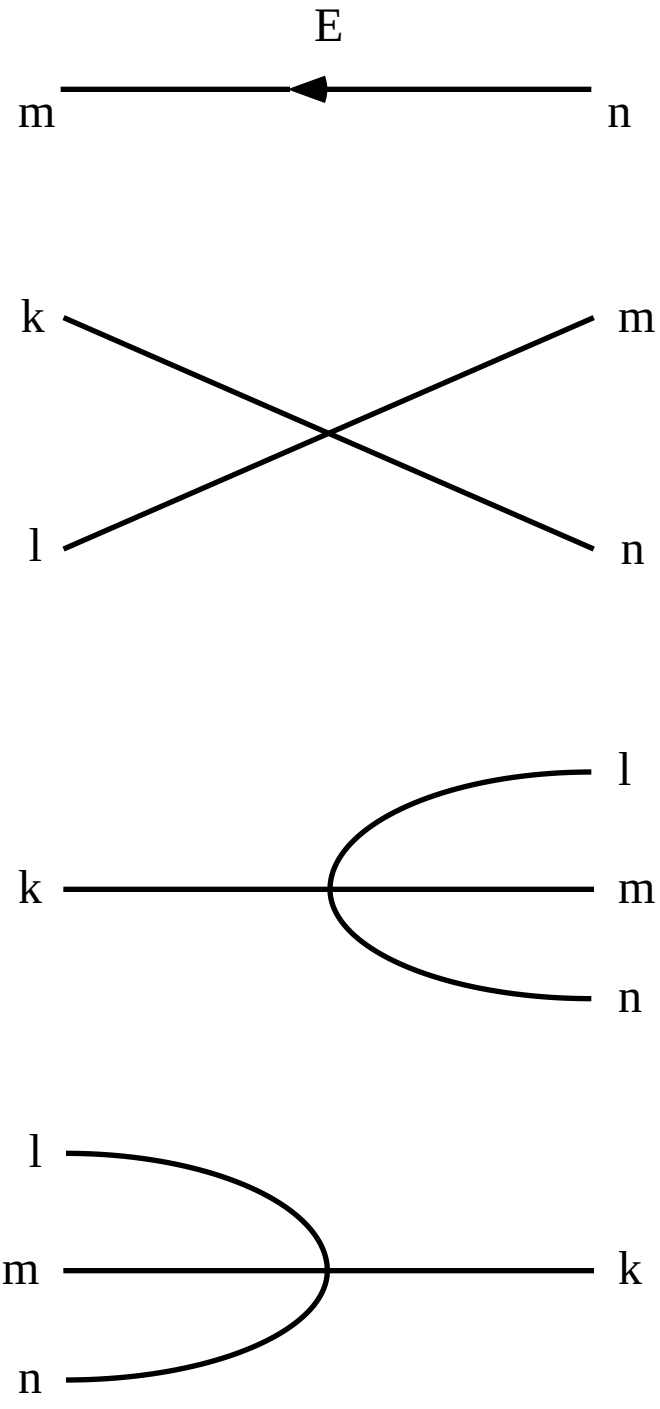
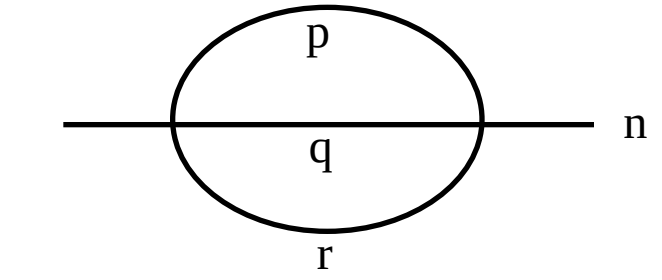
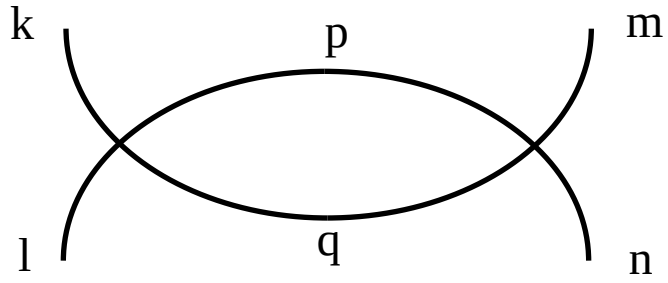

E
m
n
$\frac{i}{E - \omega_n + i\varepsilon} \delta_{m,n}$
k
m
l
n
$- i \lambda \delta_{k+l, m+n}$
l
k
m
n
$- i g \delta_{k, l+m+n}$
l
m
k
n

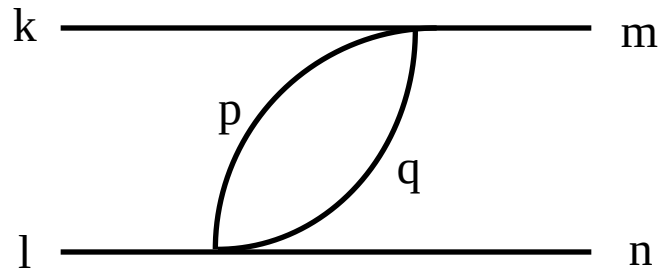
FIG. 1



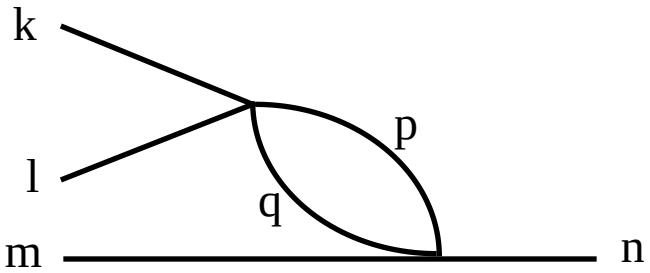
(a)



(b1)



(b2)



(c)

FIG. 2

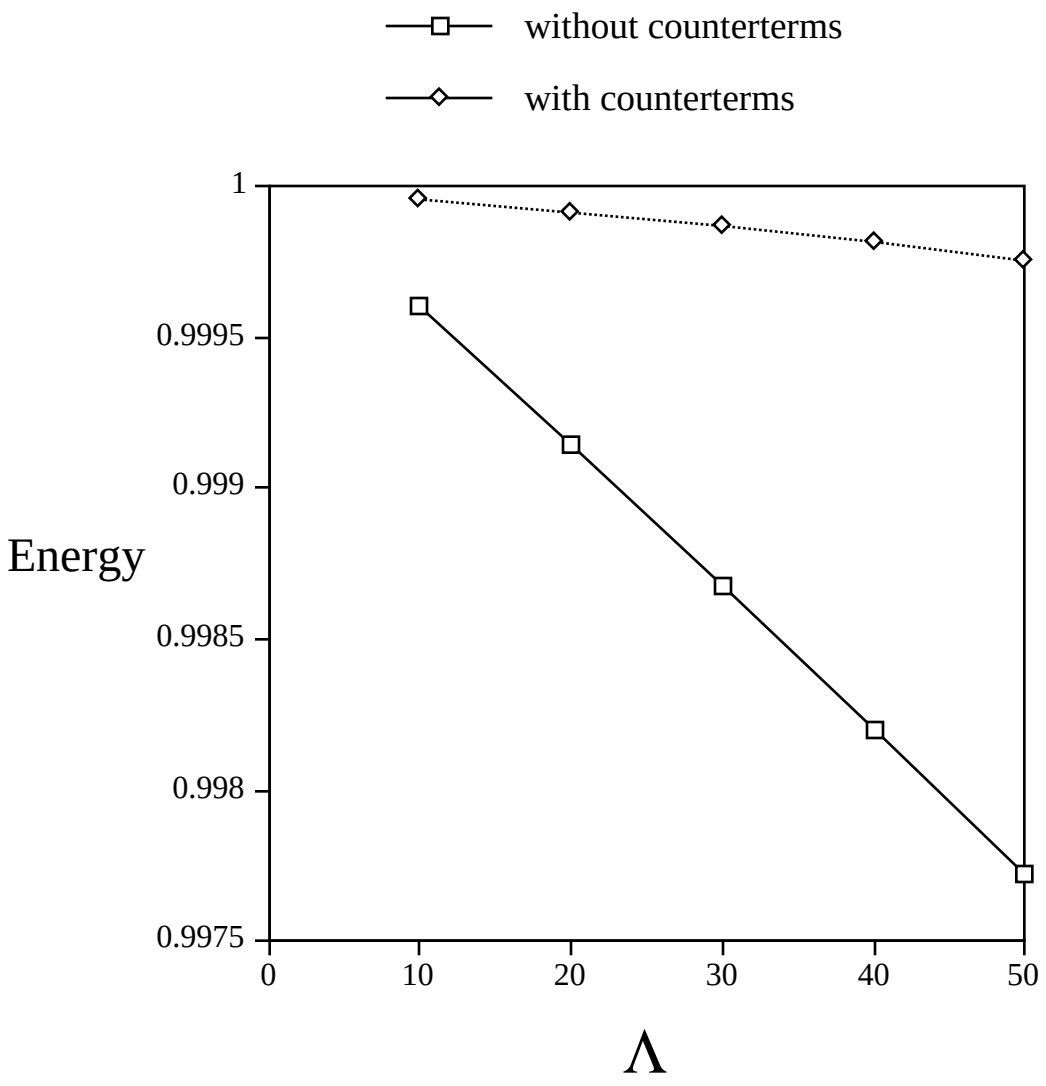

without counterterms
with counterterms
1
0.9995
0.999
0.9985
0.998
0.9975
Energy
0
10
20
30
40
50
Λ

FIG. 3-a

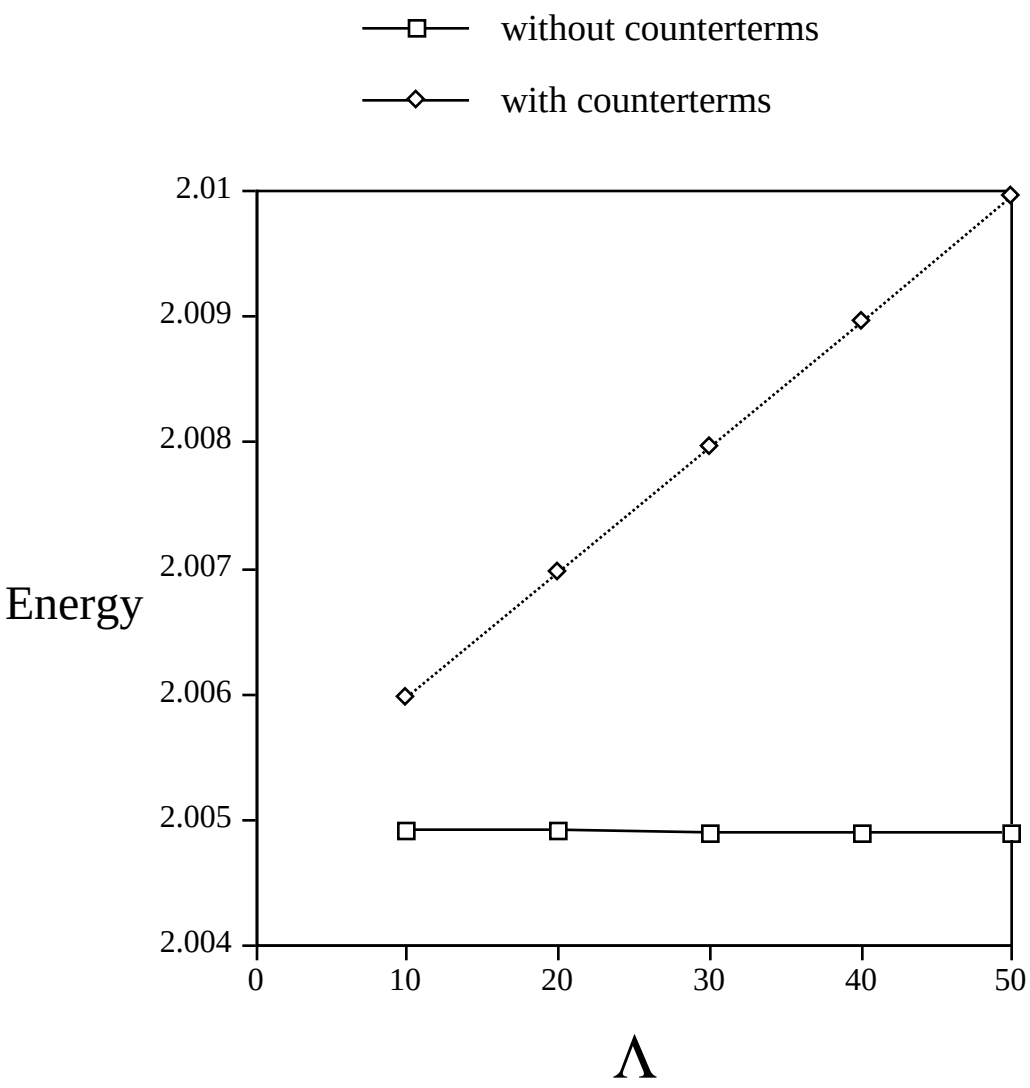
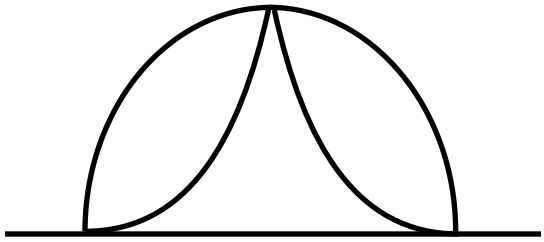

without counterterms
with counterterms
Energy
2.01
2.009
2.008
2.007
2.006
2.005
2.004
0
10
20
30
40
50
Λ

FIG. 3-b



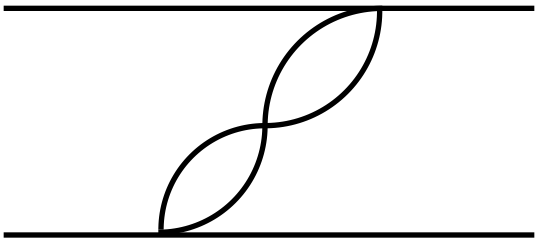


FIG. 4

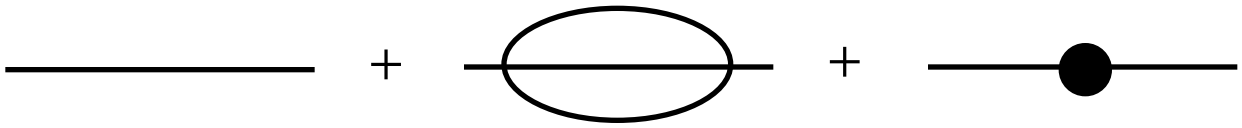


FIG. 5

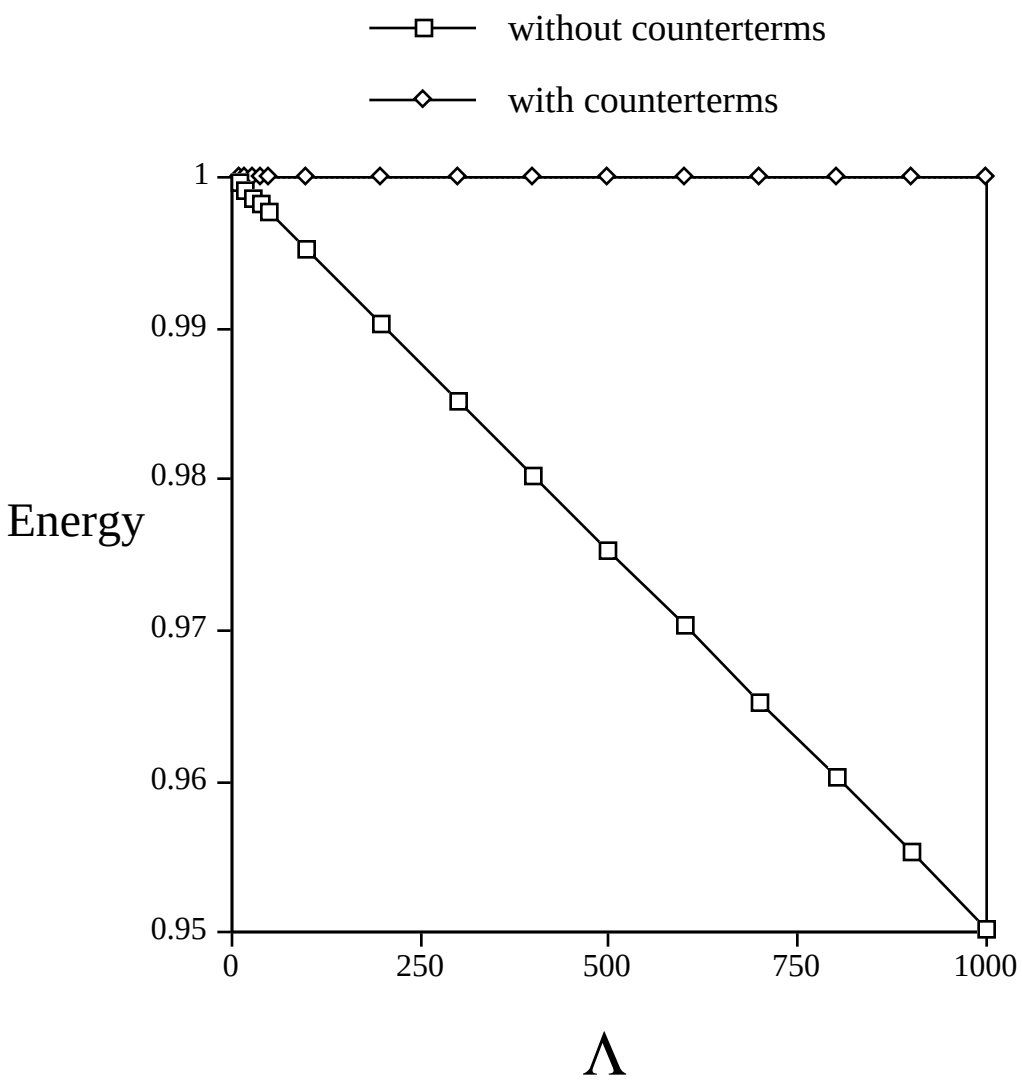

without counterterms
with counterterms
1
0.99
0.98
0.97
0.96
0.95
Energy
0
250
500
750
1000
Λ

FIG. 6

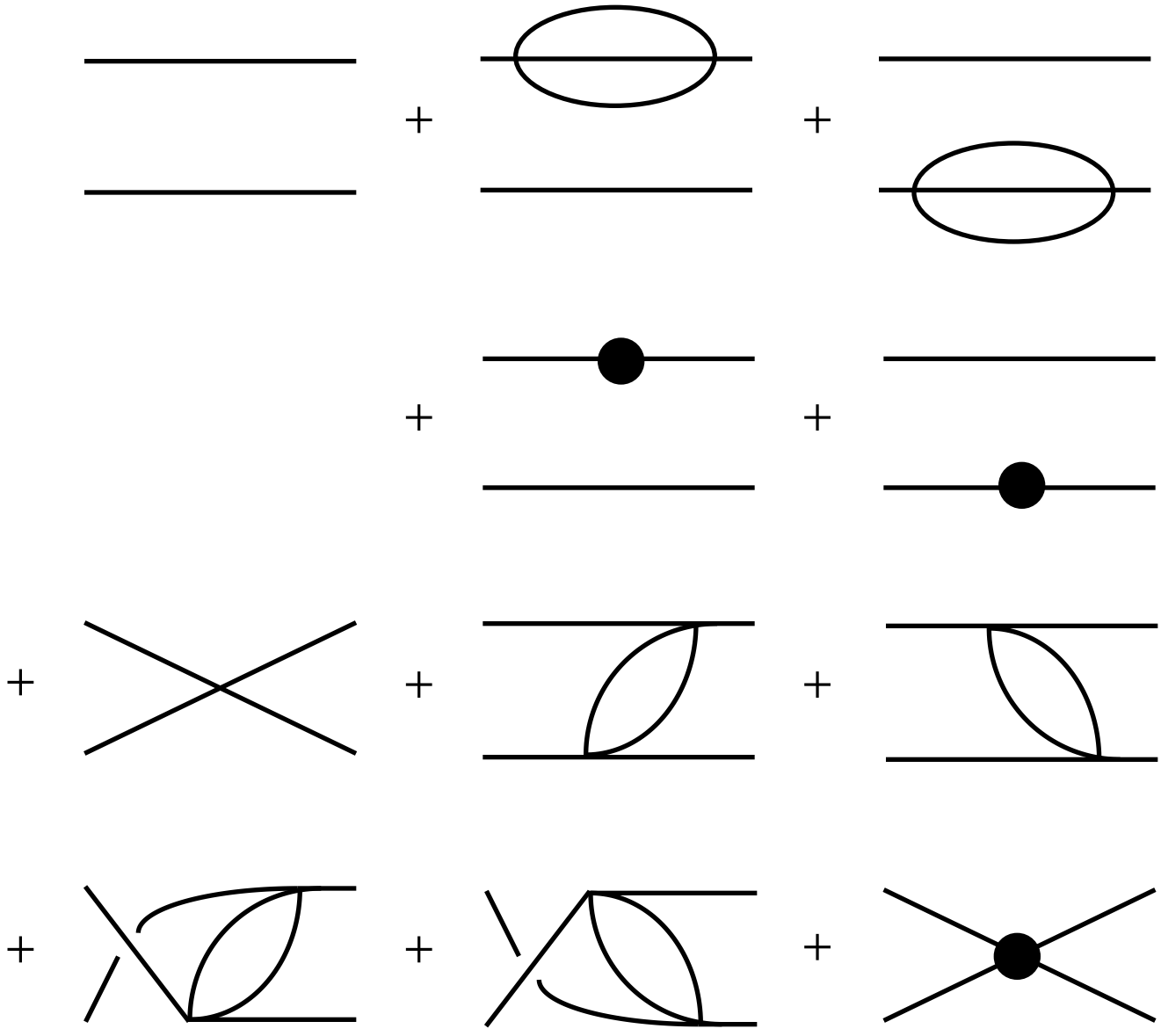


FIG. 7

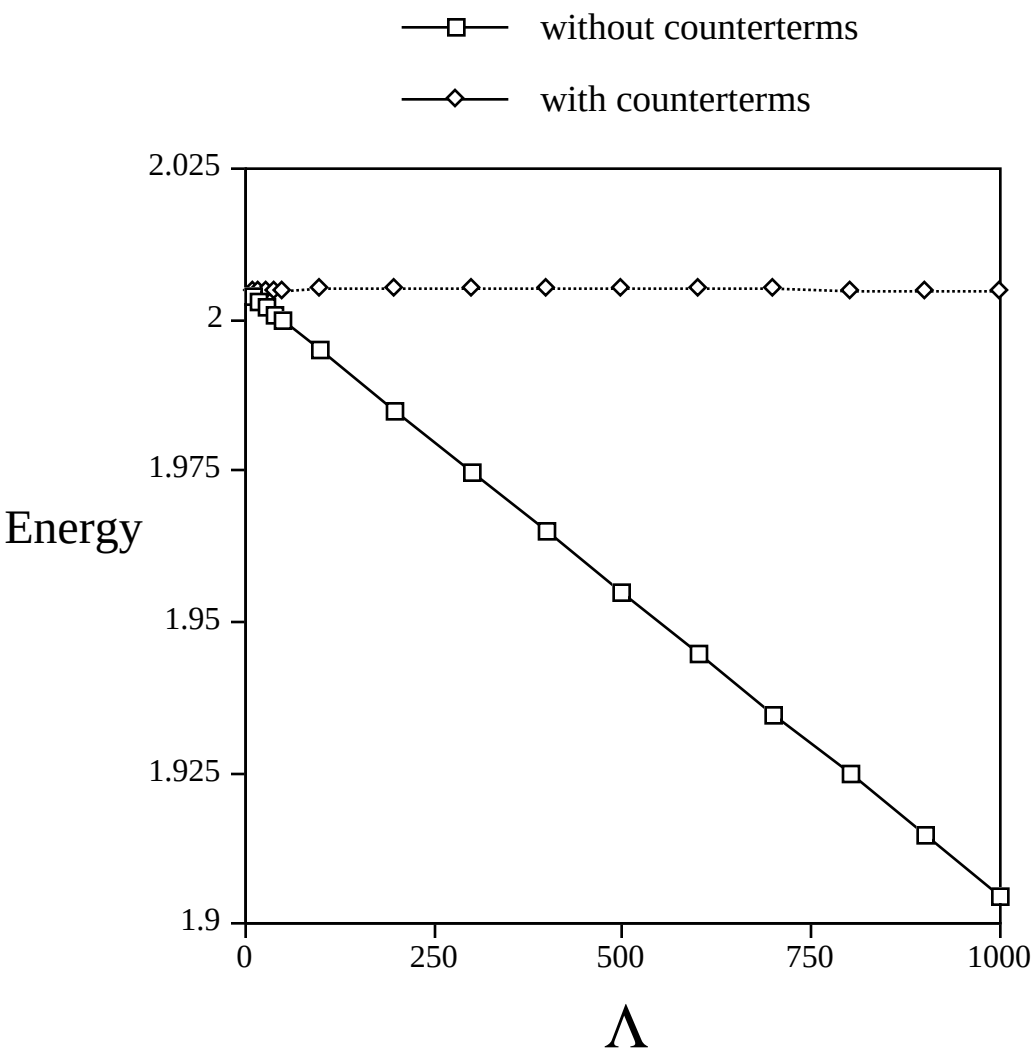
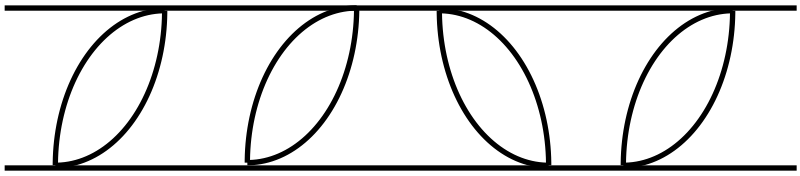

without counterterms
with counterterms
2.025
2
1.975
1.95
1.925
1.9
Energy
0
250
500
750
1000
Λ

FIG. 8

(a)



(b)

FIG. 9